

# ACTOR OF AN ALTERNATIVE ALGEBRA

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ABSTRACT. We define a category  $\mathbf{gAlt}$  of  $\mathbf{g}$ -alternative algebras over a field  $F$  and present the category of alternative algebras  $\mathbf{Alt}$  as a full subcategory of  $\mathbf{gAlt}$ ; in the case  $\text{char } F \neq 2$ , we have  $\mathbf{Alt} = \mathbf{gAlt}$ . For any  $\mathbf{g}$ -alternative algebra  $A$  we give a construction of a universal strict general actor  $\mathfrak{B}(A)$  of  $A$ . We define the subset  $\text{Asoci}(A)$  of  $A$ , and show that it is a  $\mathfrak{B}(A)$ -substructure of  $A$ . We prove that if  $\text{Asoci}(A) = 0$ , then there exists an actor of  $A$  in  $\mathbf{gAlt}$  and  $\text{Act}(A) = \mathfrak{B}(A)$ . In particular, we obtain that if  $A$  is anticommutative and  $\text{Ann}(A) = 0$ , then there exists an actor of  $A$  in  $\mathbf{gAlt}$ ; from this, under the same conditions, we deduce the existence of an actor in  $\mathbf{Alt}$ .

## 1. INTRODUCTION

The paper is dedicated to the problem of the existence of universal acting objects - actors in the category of alternative algebras (see Section 2 for the definitions). Actions in algebraic categories were studied in [11], [17], [14], [16], [18], [15], [13] and in other papers. The authors were looking for the analogs of the group of automorphisms of a group for associative algebras, rings, commutative associative algebras, Lie algebras, crossed modules and Leibniz algebras. They gave the constructions of universal objects (universal in different senses) in the corresponding categories and studied their properties. These objects are: the ring of bimultiplications of a ring, the associative algebras of multiplications (or bimultipliers) of an associative algebra and multiplications (or multipliers) of a commutative associative algebra, the actor of a crossed module, the Lie algebra of derivations and the Leibniz algebra of biderivations of a Lie and a Leibniz algebras, respectively. In [6] was proposed a categorical approach to this problem, which was continued in [1], [2], [4], [5] and [3]. In particular, representable internal object action was defined and necessary and sufficient conditions of its existence in a semi-abelian category was established. In [8] for any category of interest  $\mathbb{C}$ , in the sense of [19], we defined the corresponding category of groups with operations  $\mathbb{C}_G$ ,  $\mathbb{C} \subseteq \mathbb{C}_G$ ; for any object  $A \in \mathbb{C}$  we defined an actor  $\text{Act}(A)$  of  $A$ ; this notion is equivalent to the one of split extension classifier for  $A$ , defined in a semi-abelian category in [1]. In the same paper we gave a definition and a construction of a universal strict general actor  $\mathfrak{B}(A)$  of  $A$ , which is an object in  $\mathbb{C}_G$  in general and has all universal properties of the objects listed above. We proved that there exists an actor of  $A$  in  $\mathbb{C}$  if and only if the semidirect product  $\mathfrak{B}(A) \ltimes A$  is an object in  $\mathbb{C}$  and, in this case,  $\mathfrak{B}(A)$  is an actor of  $A$  in  $\mathbb{C}$ . Applying this result we considered examples of groups,

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associative algebras, Lie and Leibniz algebras, crossed modules and precrossed modules. In [9] we gave the construction of an actor of a precrossed module, where we introduced the notion of a generalized Whitehead's derivation. In the present paper we define a category of general alternative (g-alternative) algebras over a field  $F$  (denoted by  $\mathbf{gAlt}$ ), which is a category of interest. We present the category of alternative algebras (denoted by  $\mathbf{Alt}$ ) as a full subcategory of  $\mathbf{gAlt}$ . In the case, where  $\text{char } F \neq 2$ , we have  $\mathbf{Alt} = \mathbf{gAlt}$ . Applying the results of [8], for any g-alternative algebra  $A$  we give a construction of a universal strict general actor of  $A$  and obtain sufficient conditions for the existence of an actor of  $A$  in  $\mathbf{gAlt}$ . From this we easily deduce analogous results for the case of alternative algebras.

In Section 2 we recall the definitions of a category of interest, the general category of groups with operations of a category of interest, and give examples. We recall the definitions of structure, derived action and actor in a category of interest. We state a necessary and sufficient condition for the existence of an actor in a category of interest in terms of a universal strict general actor. The construction of this object for the case of the category  $\mathbf{gAlt}$  is given in Section 3. In the beginning of this section we define derived actions in the categories of g-alternative and alternative algebras. We state some properties of g-alternative algebras in terms of the axioms of the corresponding category of interest, which are essentially well-known for the case of alternative algebras. Then we give a construction of a universal strict general actor  $\mathfrak{B}(A)$  of a g-alternative algebra  $A$ ; as special cases of known definitions we define a semidirect product and the center of an object in  $\mathbf{gAlt}$  (equivalently, in  $\mathbf{Alt}$ ). In Section 4 for any g-alternative algebra  $A$  we define the  $\mathfrak{B}(A)$ -substructure  $\text{Soci}(A)$  of  $A$ , due to which we define certain subset  $\text{Asoci}(A)$ , which turned out to be a  $\mathfrak{B}(A)$ -substructure of  $A$ . We study the properties of this object, which are applied in what follows. In Section 5 we give an example of algebra, which shows that the object  $\mathfrak{B}(A)$  is not a g-alternative algebra in general; we state sufficient conditions for  $\mathfrak{B}(A) \in \mathbf{gAlt}$ . In Section 6 we give examples of algebras for which the action of  $\mathfrak{B}(A)$  on  $A$  is not a derived action in general. The final result states that if  $\text{Asoci}(A) = 0$ , then  $\mathfrak{B}(A)$  is a g-alternative algebra, the action of  $\mathfrak{B}(A)$  on  $A$  is a derived action in  $\mathbf{gAlt}$  and according to the general result, given in [9],  $\mathfrak{B}(A) = \text{Act}(A)$ . In particular, if  $A$  is anticommutative (i.e.  $aa' = -a'a$ , for any  $a, a' \in A$ ) and  $\text{Ann}(A) = 0$ , then there exists an actor of  $A$  in  $\mathbf{gAlt}$ . At the end of the section we give another description of an actor of a g-alternative algebra in terms of a certain algebra of bimultiplications, defined in analogous way as for the case of rings or associative algebras. In Section 7 we investigate more properties of the object  $\mathfrak{B}(A)$  under the conditions that  $A$  is anticommutative with  $\text{Ann}(A) = 0$ ; in particular, it is proved that  $\mathfrak{B}(A) \in \mathbf{Alt}$  and its action on  $A$  is a derived action in  $\mathbf{Alt}$ , which give the sufficient conditions for the existence of an actor in the category of alternative algebras. Note that in the cases of associative algebras and rings the algebras of bimultiplications played an important role in the extension and obstruction theories [11, 17]. From this point of view the results obtained in this paper could be applied to a cohomology and the corresponding extension and obstruction theories of alternative algebras.

## 2. PRELIMINARY DEFINITIONS AND RESULTS

This section contains some well-known and new definitions and results which will be used in what follows.

Let  $\mathbb{C}$  be a category of groups with a set of operations  $\Omega$  and with a set of identities  $\mathbb{E}$ , such that  $\mathbb{E}$  includes the group identities and the following conditions hold. If  $\Omega_i$  is the set of  $i$ -ary operations in  $\Omega$ , then:

- (a)  $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ ;
- (b) the group operations (written additively:  $(0, -, +)$  are elements of  $\Omega_0$ ,  $\Omega_1$  and  $\Omega_2$  respectively. Let  $\Omega'_2 = \Omega_2 \setminus \{+\}$ ,  $\Omega'_1 = \Omega_1 \setminus \{-\}$  and assume that if  $*$   $\in \Omega_2$ , then  $\Omega'_2$  contains  $*^\circ$  defined by  $x *^\circ y = y * x$ . Assume further that  $\Omega_0 = \{0\}$ ;
- (c) for each  $*$   $\in \Omega'_2$ ,  $\mathbb{E}$  includes the identity  $x * (y + z) = x * y + x * z$ ;
- (d) for each  $\omega \in \Omega'_1$  and  $*$   $\in \Omega'_2$ ,  $\mathbb{E}$  includes the identities  $\omega(x + y) = \omega(x) + \omega(y)$  and  $\omega(x) * y = \omega(x * y)$ .

Note that the group operation is denoted additively, but it is not commutative in general. A category  $\mathbb{C}$  defined above is called a *category of groups with operations*. The idea of the definition comes from [10] and axioms are from [19] and [20]. We formulate two more axioms on  $\mathbb{C}$  (Axiom (7) and Axiom (8) in [19]).

If  $C$  is an object of  $\mathbb{C}$  and  $x_1, x_2, x_3 \in C$ :

*Axiom 1.*  $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$ , for each  $*$   $\in \Omega'_2$ .

*Axiom 2.* For each ordered pair  $(*, \bar{*}) \in \Omega'_2 \times \Omega'_2$  there is a word  $W$  such that

$$\begin{aligned} (x_1 * x_2) \bar{*} x_3 &= W(x_1(x_2 x_3), x_1(x_3 x_2), (x_2 x_3) x_1, \\ &\quad (x_3 x_2) x_1, x_2(x_1 x_3), x_2(x_3 x_1), (x_1 x_3) x_2, (x_3 x_1) x_2), \end{aligned}$$

where each juxtaposition represents an operation in  $\Omega'_2$ .

A category of groups with operations satisfying Axiom 1 and Axiom 2 is called a *category of interest* in [19].

Denote by  $\mathbb{E}_G$  the subset of identities of  $\mathbb{E}$  which includes the group identities and the identities (c) and (d). We denote by  $\mathbb{C}_G$  the corresponding category of groups with operations. Thus we have  $\mathbb{E}_G \hookrightarrow \mathbb{E}$ ,  $\mathbb{C} = (\Omega, \mathbb{E})$ ,  $\mathbb{C}_G = (\Omega, \mathbb{E}_G)$  and there is a full inclusion functor  $\mathbb{C} \hookrightarrow \mathbb{C}_G$ . The category  $\mathbb{C}_G$  is called a *general category of groups with operations* of a category of interest  $\mathbb{C}$  in [7] and [8].

**Examples of categories of interest.** In the case of the category of associative algebras with multiplication represented by  $*$ , we have  $\Omega'_2 = \{*, *^\circ\}$ . For Lie algebras take  $\Omega'_2 = ([, ], [, ]^\circ)$  (where  $[a, b]^\circ = [b, a] = -[a, b]$ ). For Leibniz algebras (see [15]), take  $\Omega'_2 = ([, ], [, ]^\circ)$ , (here  $[a, b]^\circ = [b, a]$ ). It is easy to see that the categories of all these algebras are categories of interest. In the cases of the categories of groups, abelian groups and modules over a ring we have  $\Omega'_2 = \emptyset$ . As it is noted in [19] Jordan algebras do not satisfy Axiom 2. Below we will see that the category of alternative algebras is a category of interest as well.

**Definition 2.1.** [19] Let  $A, B \in \mathbb{C}$ . An *extension* of  $B$  by  $A$  is a sequence

$$(2.1) \quad 0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0$$

in which  $p$  is surjective and  $i$  is the kernel of  $p$ . We say that an extension is *split* if there is a morphism  $s: B \longrightarrow E$  such that  $ps = 1_B$ .

**Definition 2.2.** [19] A split extension of  $B$  by  $A$  is called a *B-structure* on  $A$ .

According to [19], for  $A, B \in \mathbb{C}$  “a set of actions of  $B$  on  $A$ ”, means that there is a map  $f_*: B \times A \longrightarrow A$ , for each  $*$   $\in \Omega_2$ . Instead of “set of actions” often is used for simplicity “an action”, and it means that there is a set of actions  $\{f_*\}_{* \in \Omega_2}$ . Note that a set of actions has a different meaning in [1].

A  $B$ -structure on  $A$  induces a set of actions of  $B$  on  $A$  corresponding to the operations in  $\mathbb{C}$ . If (2.1) is a split extension, then for  $b \in B$ ,  $a \in A$  and  $*$   $\in \Omega_2'$  we have

$$(2.2) \quad b \cdot a = s(b) + a - s(b),$$

$$(2.3) \quad b * a = s(b) * a.$$

Actions defined by (2.2) and (2.3) are called *derived actions* of  $B$  on  $A$  in [19]. Under *B-substructure* of  $A$  naturally we will mean a subobject  $A'$  of  $A$  which is closed under all derived actions of  $B$  on  $A'$ , i.e. left and right derived actions.

In the case of associative algebras over a ring  $R$  a derived action of  $B$  on  $A$  is a pair of  $R$ -bilinear maps

$$(2.4) \quad B \times A \longrightarrow A, \quad A \times B \longrightarrow A$$

which we denote respectively as  $(b, a) \mapsto ba$ ,  $(a, b) \mapsto ab$ , with conditions

$$\begin{aligned} (b_1 b_2) a &= b_1 (b_2 a), \\ a (b_1 b_2) &= (a b_1) b_2, \\ (b_1 a) b_2 &= b_1 (a b_2), \\ (a_1 a_2) b &= a_1 (a_2 b), \\ b (a_1 a_2) &= (b a_1) a_2, \\ (a_1 b) a_2 &= a_1 (b a_2), \end{aligned}$$

for any  $a_1, a_2 \in A$  and  $b_1, b_2 \in B$ . Note that these identities are well-known, and they can be obtained easily from the definition of a derived action (2.3) and the associativity axiom.

According to [19], in any category of interest, given a set of actions of  $B$  on  $A$ , the *semidirect product*  $B \ltimes A$  is the universal algebra, whose underlying set is  $B \times A$  and the operations are defined by

$$\begin{aligned} (b', a') + (b, a) &= (b' + b, a' + b' \cdot a), \\ (b', a') * (b, a) &= (b' * b, a' * a + a' * b + b' * a). \end{aligned}$$

**Theorem 2.3.** [19] *A set of actions of  $B$  on  $A$  is a set of derived actions if and only if  $B \ltimes A$  is an object of  $\mathbb{C}$ .*

**Definition 2.4.** [7, 8] For any object  $A$  in  $\mathbb{C}$ , an actor of  $A$  is an object  $\text{Act}(A)$  in  $\mathbb{C}$ , which has a derived action on  $A$  in  $\mathbb{C}$ , and for any object  $C$  of  $\mathbb{C}$  and a derived action of  $C$  on  $A$  there is a unique morphism  $\varphi: C \longrightarrow \text{Act}(A)$  with  $c \cdot a = \varphi(c) \cdot a$ ,  $c * a = \varphi(c) * a$  for any  $* \in \Omega_2'$ ,  $a \in A$  and  $c \in C$ .

In [8], for any object  $A$  of a category of interest  $\mathbb{C}$ , we define a universal strict general actor  $\mathfrak{B}(A)$  of  $A$ , which is an object of  $\mathbb{C}_G$ , and give the corresponding construction. We present this construction for the case of  $g$ -alternative algebras (see below the definition) in Section 3.

**Theorem 2.5.** [8] *Let  $\mathbb{C}$  be a category of interest and  $A \in \mathbb{C}$ .  $A$  has an actor if and only if the semidirect product  $\mathfrak{B}(A) \ltimes A$  is an object of  $\mathbb{C}$ . If it is the case, then  $\text{Act}(A) = \mathfrak{B}(A)$ .*

From Theorems 2.3 and 2.5 we have

**Corollary 2.6.** *An object  $A$  of a category of interest  $\mathbb{C}$  has an actor if and only if  $\mathfrak{B}(A) \in \mathbb{C}$  and the action of  $\mathfrak{B}(A)$  on  $A$  is a derived action.*

Recall that an *alternative algebra*  $A$  over a field  $F$  is an algebra which satisfies the identities  $x^2y = x(xy)$  and  $yx^2 = (yx)x$ , for all  $x, y \in A$ . These identities are known respectively as the left and right alternative laws. We denote the corresponding category of alternative algebras by  $\text{Alt}$ . Clearly any associative algebra is alternative. The class of 8-dimensional Cayley algebras is an important class of alternative algebras which are not associative [21]; commutative alternative algebras are Jordan algebras.

Note that in all categories of algebras, considered in this paper as categories of interest, algebras are generally without unit, but, of course, algebras with unit are also included in these categories. We will always note when we deal with algebras over a field with characteristic 2, but nevertheless in such cases we will use the sign “ $-$ ” before the elements of the algebra, in order to denote the additive inverse elements, e.g.  $-a$ , for the element inverse to  $a$  of the algebra  $A$ ; this we do essentially in order not to confuse a reader during the computations while we apply different axioms or certain assumptions on algebras.

Here we introduce the notion of a general alternative algebra.

**Definition 2.7.** A *general alternative algebra* (shortly  *$g$ -alternative algebra*)  $A$  over a field  $F$  is an algebra, which satisfies the following two axioms for any  $x, y, z \in A$

$$\text{Axiom } 2_1. \quad x(yz) = (xy)z + (yx)z - y(xz);$$

$$\text{Axiom } 2_2. \quad (xy)z = x(yz) + x(zx) - (xz)y.$$

These axioms are dual to each other in the sense, that if  $x \circ y = yx$ , then Axiom  $2_1$  for the operation  $\circ$  gives Axiom  $2_2$  for the original operation, and obviously, Axiom  $2_2$  for the  $\circ$  operation gives Axiom  $2_1$ . We consider these conditions as Axiom 2 and consequently, the category of  $g$ -alternative algebras can be interpreted as a category of interest, which will be denoted by  $g\text{Alt}$ . According to the definition of a general category of groups with operations for a given category of interest, we obtain that  $\text{Alt}_G = g\text{Alt}_G$  and it is a category of groups together with the multiplication operation, which satisfies the identity  $x(y + z) = xy + xz$ .

Denote by  $\overline{\text{gAlt}}$  the full subcategory of  $\text{gAlt}$  of those objects and homomorphisms between them, which satisfy the following condition

$$E_1. \quad (xy)x = x(yx), \text{ for any } x, y \in A.$$

This identity is called the *flexible law* [21] or *flexible identity* [22].

**Proposition 2.8.** (i) For any field  $F$ , we have the equality  $\text{Alt} = \overline{\text{gAlt}}$ ;  
(ii) If  $\text{char } F \neq 2$ , then  $\text{gAlt} = \overline{\text{gAlt}} = \text{Alt}$ .

*Proof.* (i) Let  $A$  be an alternative algebra, then we have

$$(x + y)^2 z = (x + y)((x + y)z), \text{ for any } x, y, z \in A,$$

from which follows Axiom 2<sub>1</sub>. Analogously, the identity  $x(y + z)^2 = (x(y + z))(y + z)$  gives Axiom 2<sub>2</sub>. Thus we have  $\text{Alt} \subseteq \text{gAlt}$ . It is well-known fact that every alternative algebra satisfies the condition  $E_1$ ; see the equivalent definition of alternative algebras and the proof of Artin's theorem, e.g. in [12]. We include here the proof based on the fact that alternative algebras are g-alternative algebras. From Axiom 2<sub>1</sub>, for  $y = z$  we have

$$(2.5) \quad xy^2 = (xy)y + (yx)y - y(xy)$$

and since  $xy^2 = (xy)y$ , we obtain  $(yx)y = y(xy)$ , which is the condition  $E_1$ . Now we shall show that  $\overline{\text{gAlt}} \subseteq \text{Alt}$ . Let  $A \in \overline{\text{gAlt}}$ , again from (2.5), applying the condition  $E_1$  we obtain  $xy^2 = (xy)y$ . Analogously, from Axiom 2<sub>2</sub> for  $x = y$  we have  $x^2 z = x(xz) + x(zx) - (xz)x$ ; this by  $E_1$  implies  $x^2 z = x(xz)$ , which ends the proof of (i).

(ii) We have only to show that if  $\text{char } F \neq 2$ , then every g-alternative algebra satisfies the condition  $E_1$ . From Axiom 2<sub>2</sub> for  $y = z$ , we have  $(xy)y = xy^2 + xy^2 - (xy)y$ . From this and (2.5) we have  $2y(xy) = 2(yx)y$ , applying the fact that  $\text{char } F \neq 2$  we obtain the condition  $E_1$ , which ends the proof.  $\square$

The following example shows that in the case  $\text{char } F = 2$  we have the strict inclusion  $\text{Alt} \subset \text{gAlt}$ .

**Example 2.1.** Let  $A$  be the free g-alternative algebra generated by the one element set  $\{x\}$  over a field  $F$ , and  $\text{char } F = 2$ . Then, according to Axiom 2<sub>1</sub>, since  $(xx)x + (xx)x = 0$ , we only obtain that  $x(xx) = -x(xx)$ , as it is for any element  $a \in A$ , i.e.  $a = -a$ , since  $\text{char } F = 2$ . Analogously, from Axiom 2<sub>2</sub>, we will have  $(xx)x = -(xx)x$ , which shows that  $A$  is not an alternative algebra.

### 3. DERIVED ACTIONS AND UNIVERSAL ACTING OBJECTS IN THE CATEGORIES OF G-ALTERNATIVE AND ALTERNATIVE ALGEBRAS

According to the definition of derived action in a category of interest, in the category of g-alternative algebras over a field  $F$  we obtain the following: a derived action of  $B$  on  $A$  in  $\text{gAlt}$  is a pair of  $F$ -bilinear maps (2.4), which we denote by  $(b, a) \mapsto ba$ ,  $(a, b) \mapsto ab$  with

the conditions

$$\begin{aligned}
 \text{I}_1. \quad & b(a_1a_2) = (ba_1)a_2 + (a_1b)a_2 - a_1(ba_2), \\
 \text{I}_2. \quad & (a_1a_2)b = a_1(a_2b) + a_1(ba_2) - (a_1b)a_2, \\
 \text{I}_3. \quad & (ba_1)a_2 = b(a_1a_2) + b(a_2a_1) - (ba_2)a_1, \\
 \text{I}_4. \quad & a_1(a_2b) = (a_1a_2)b + (a_2a_1)b - a_2(a_1b), \\
 \text{II}_1. \quad & (b_1b_2)a = b_1(b_2a) + b_1(ab_2) - (b_1a)b_2, \\
 \text{II}_2. \quad & a(b_1b_2) = (ab_1)b_2 + (b_1a)b_2 - b_1(ab_2), \\
 \text{II}_3. \quad & (ab_1)b_2 = a(b_1b_2) + a(b_2b_1) - (ab_2)b_1, \\
 \text{II}_4. \quad & b_1(b_2a) = (b_1b_2)a + (b_2b_1)a - b_2(b_1a),
 \end{aligned}$$

for any  $a_1, a_2 \in A$  and  $b_1, b_2 \in B$ . These identities are obtained according to (2.3) and Axiom 2 for g-alternative algebras.

For the case of alternative algebras for derived actions we will have  $\text{I}_1 - \text{I}_4, \text{II}_1 - \text{II}_4$ , and two more identities obtained from the condition  $\text{E}_1$ :

$$\begin{aligned}
 \text{III}_1. \quad & a(ba) = (ab)a, \\
 \text{III}_2. \quad & b(ab) = (ba)b,
 \end{aligned}$$

for any  $a \in A, b \in B$ . In the case where  $A$  is a vector space over a field  $F$  we obtain the well-known definition of an alternative bimodule or, equivalently, of a representation of an alternative algebra [21].

From Axiom 2 we easily deduce the following useful identities in terms of Axiom 2 of the corresponding category of interest, which express the well-known properties of alternative algebras (see e.g. [21], [22]). We shall use the notation like  $(1 \leftrightarrow 2, -)$  in order to denote that right side of the identity is obtained by the permutation of the first and the second elements from the left side, and the signs of the summands are changed; the notation like  $(1 \rightarrow 3 \rightarrow 2 \rightarrow 1)$  denotes that the right side of the identity is obtained by the following changes in the left side: the first element takes the place of the third one, the third element the place of the second one and the second element takes the place of the first one.

$$\begin{aligned}
 (3.1) \quad & (1 \leftrightarrow 2, -) & (xy)z - x(yz) &= -(yx)z + y(xz) \\
 (3.2) \quad & (1 \leftrightarrow 3, -) & (xy)z - x(yz) &= -(zy)x + z(yx) \\
 (3.3) \quad & (2 \leftrightarrow 3, -) & (xy)z - x(yz) &= -(xz)y + x(zy) \\
 (3.4) \quad & (1 \rightarrow 3 \rightarrow 2 \rightarrow 1) & (xy)z - x(yz) &= (yz)x - y(zx) \\
 (3.5) \quad & (1 \rightarrow 2 \rightarrow 3 \rightarrow 1) & (xy)z - x(yz) &= (zx)y - z(xy) \\
 (3.6) \quad & & (xy)z + z(xy) &= x(yz) + (zx)y \\
 (3.7) \quad & & (xy)z + (yx)z &= x(yz) + y(xz) \\
 (3.8) \quad & & z(xy) + z(yx) &= (zx)y + (zy)x
 \end{aligned}$$

for any  $x, y, z \in A$ . Note that we have  $(3.1) \Leftrightarrow \text{Axiom } 2_1$ ,  $(3.3) \Leftrightarrow \text{Axiom } 2_1$ , and  $(3.4)$  from right to the left is the same as  $(3.5)$ . In the case  $\text{char } F \neq 2$ ,  $(3.2)$  is equivalent to the flexible law  $E_1$ .

Let  $\{B_i\}_{i \in I}$  be the set of all g-alternative algebras, which have a derived action on  $A$  in  $\text{gAlt}$ . From the properties  $I_1 - I_4$  it follows that identities  $(3.1) - (3.8)$  are true for any  $x, y, z \in A$ , where one of the elements from  $x, y, z$  is in  $B_i$  for any  $i \in I$ .

Let  $A$  and  $B$  be g-alternative (resp. alternative) algebras and  $B$  has a derived action on  $A$  in  $\text{gAlt}$  (resp.  $\text{Alt}$ ). According to the general definition of a semidirect product in a category of interest given in Section 2,  $B \ltimes A$  is a g-alternative (resp. an alternative) algebra, whose underlying set is  $B \times A$  and whose operations are given by

$$\begin{aligned} (b', a') + (b, a) &= (b' + b, a' + a), \\ (b', a')(b, a) &= (b'b, a'a + a'b + b'a). \end{aligned}$$

According to Definition 2.4, for any g-alternative (resp. alternative) algebra  $A$ , an *actor* of  $A$  is an object  $\text{Act}(A) \in \text{gAlt}$  (resp.  $\text{Act}(A) \in \text{Alt}$ ), which has a derived action on  $A$  in  $\text{gAlt}$  (resp.  $\text{Alt}$ ) and for any g-alternative (resp. alternative) algebra  $C$  and a derived action of  $C$  on  $A$ , there is a unique homomorphism  $\varphi: C \rightarrow \text{Act}(A)$  with  $ca = \varphi(c)a$ , for any  $a \in A$  and  $c \in C$ .

Here we give a construction of a universal strict general actor  $\mathfrak{B}(A)$  of a g-alternative algebra  $A$ , which is a special case of the construction given in [8] for categories of interest. In this case we have only two binary operations: the addition, denoted by “+”, and the multiplication, which will be denoted here by dot “ $\cdot$ ”. Note that this sign was omitted usually in above expressions for g-alternative algebras, and it will be so in the next sections as well, when there is no confusion. Since the addition is commutative, the action corresponding to this operation is trivial. Thus we will deal only with actions, which are defined by multiplication according to (2.3), and this action will be denoted by  $\cdot$  as well

$$b \cdot a = s(b) \cdot a.$$

Consider all split extensions of  $A$  in  $\text{gAlt}$

$$E_j: 0 \longrightarrow A \xrightarrow{i_j} C_j \xrightarrow{p_j} B_j \longrightarrow 0, \quad j \in \mathbb{J}.$$

Note that it may happen that  $B_j = B_k = B$ , for  $j \neq k$ , in this case the corresponding extensions derive different actions of  $B$  on  $A$ . Let  $\{(b_j \cdot, \cdot b_j) | b_j \in B_j\}$  be the set of all pairs of  $F$ -linear maps  $A \rightarrow A$ , defined by the action of  $B_j$  on  $A$ . For any element  $b_j \in B_j$  denote  $\mathbf{b}_j = (b_j \cdot, \cdot b_j)$ . Let  $\mathbb{B} = \{\mathbf{b}_j | b_j \in B_j, j \in \mathbb{J}\}$ .

According to Axiom 2 from the definition of  $\text{gAlt}$  as a category of interest, we define the multiplication,  $\mathbf{b}_i \cdot \mathbf{b}_k$ , for the elements of  $\mathbb{B}$  by the equalities

$$\begin{aligned} (\mathbf{b}_i \cdot \mathbf{b}_k) \cdot (a) &= b_i \cdot (b_k \cdot a) + b_i \cdot (a \cdot b_k) - (b_i \cdot a) \cdot b_k, \\ (a) \cdot (\mathbf{b}_i \cdot \mathbf{b}_k) &= (a \cdot b_i) \cdot b_k + (b_i \cdot a) \cdot b_k - b_i \cdot (a \cdot b_k). \end{aligned}$$

For  $b = \mathbf{b}_{i_1} \cdot \dots \cdot \mathbf{b}_{i_n}$  with certain brackets,  $b \cdot a$  is defined in a natural way step by step according to brackets and to above defined equalities.



We define the operation of addition by

$$(\mathbf{b}_i + \mathbf{b}_k) \cdot (a) = b_i \cdot a + b_k \cdot a.$$

For the unary operation “-” we define

$$\begin{aligned} (-\mathbf{b}_k) \cdot (a) &= -(b_k \cdot a), \\ (-b) \cdot (a) &= -(b \cdot (a)), \\ -(b_1 + \dots + b_n) &= -b_n - \dots - b_1, \end{aligned}$$

where  $b, b_1, \dots, b_n$  are certain combinations of the dot operation on the elements of  $\mathbb{B}$ , i.e. the elements of the type  $\mathbf{b}_{i_1} \cdot \dots \cdot \mathbf{b}_{i_n}$ , where we mean certain brackets,  $n > 1$ . Obviously, the addition is commutative.

Denote by  $\mathfrak{B}'(A)$  the set of all pairs of  $F$ -linear maps obtained by performing all kinds of above defined operations on the elements of  $\mathbb{B}$ . We define the following relation: we will write  $b \sim b'$ , for  $b, b' \in \mathfrak{B}'(A)$ , if and only if  $b \cdot (a) = b' \cdot (a)$ , for any  $a \in A$ . This relation is a congruence relation on  $\mathfrak{B}'(A)$ , i.e. it is compatible with the operations we have defined in  $\mathfrak{B}'(A)$ . We define  $\mathfrak{B}(A) = \mathfrak{B}'(A) / \sim$ . The operations defined on  $\mathfrak{B}'(A)$  induce the corresponding operations on  $\mathfrak{B}(A)$ . For simplicity we will denote the elements of  $\mathfrak{B}(A)$  by the same letters  $b, b'$ , etc. instead of the classes  $clb, clb'$ , etc.

According to the general result [8, Proposition 4.1]  $\mathfrak{B}(A)$  is an object of  $\mathbf{gAlt}_G$ ; moreover, it is obvious, that  $\mathfrak{B}(A)$  is an  $F$ -algebra.

Define the set of actions of  $\mathfrak{B}(A)$  on  $A$  in a natural way: for  $b \in \mathfrak{B}(A)$  we define  $b \cdot a = b \cdot (a)$ . Thus if  $b = \mathbf{b}_{i_1} \cdot \dots \cdot \mathbf{b}_{i_n}$ , where we mean certain brackets, we have

$$b \cdot a = (\mathbf{b}_{i_1} \cdot \dots \cdot \mathbf{b}_{i_n}) \cdot (a),$$

where the right side of the equality is defined according to the brackets and Axiom 2. For  $b_k \in B_k$ ,  $k \in \mathbb{J}$ , we have

$$\mathbf{b}_k \cdot a = \mathbf{b}_k \cdot (a) = b_k \cdot a.$$

Also

$$(b_1 + b_2 + \dots + b_n) \cdot a = b_1 \cdot (a) + \dots + b_n \cdot (a), \text{ for } b_i \in \mathfrak{B}(A), i = 1, \dots, n.$$

It is easy to check (see [8, Proposition 4.2] for the general case) that the set of actions of  $\mathfrak{B}(A)$  on  $A$  is a set of derived actions in  $\mathbf{gAlt}_G$ .

The construction of a universal strict general actor of an alternative algebra  $A$  in  $\mathbf{Alt}$  is analogous to the one of  $\mathfrak{B}(A)$ ; in this case we consider all split extensions  $E_j$  of  $A$  in  $\mathbf{Alt}$ , i.e we will deal with the family  $\{B_i\}_{i \in I}$  of all alternative algebras, which have a derived action on  $A$  in  $\mathbf{Alt}$ . The corresponding object will be denoted by  $\mathfrak{B}_{\mathbf{Alt}}(A)$ , and it is an object of  $\mathbf{Alt}_G$ .

For any  $A \in \mathbf{gAlt}$ , define the map  $d: A \longrightarrow \mathfrak{B}(A)$  by  $d(a) = \mathbf{a}$ , where  $\mathbf{a} = \{(a, \cdot a)\}$ . Thus we have by definition

$$d(a) \cdot a' = a \cdot a', \quad a' \cdot d(a) = a' \cdot a, \text{ for any } a, a' \in A.$$

According to the general results (see the case of a category of interest, [8, Lemma 4.5 and Proposition 4.6])  $d$  is a homomorphism in  $\text{gAlt}_G$  and moreover  $d: A \longrightarrow \mathfrak{B}(A)$  is a crossed module in  $\text{gAlt}_G$  with certain universal property.

From the general definition of a center in categories of interest [8], (cf. [19]), for any  $g$ -alternative algebra  $A$ , we obtain that the *center* of  $A$  is defined by

$$Z(A) = \text{Ker } d = \{z \in A \mid \text{for all } a \in A, az = za = 0\}.$$

Therefore the center in this case is a left-right annihilator of  $A$ , denoted by  $\text{Ann}(A)$  (see [17] for the case of rings).

#### 4. $\text{Soci}(A)$ , $\text{Asoci}(A)$ AND THEIR PROPERTIES

For any  $x, y, z \in A \cup \mathfrak{B}(A)$ ,  $A \in \text{Alt}$ , consider the following types of elements:

- (sac)  $x(yz) + x(z y), (yz)x + (zy)x$  ;
- (as)  $x(yz) - (xy)z$ ;
- (aas)  $x(yz) + (xy)z$ ;
- (ap)  $x(yz) + y(xz), (yz)x + (yx)z$ .

Let  $\{B_i\}_{i \in I}$  be the family of  $g$ -alternative algebras which have a derived action on  $A$ . According to the above notation we define the following sets in  $A$ .

- $S_1^{\text{sac}}$  (resp.  $S_2^{\text{sac}}$ ): the set of elements of  $A$  of the type (sac), where one element (resp. two elements) from the triple  $(x, y, z)$  is (resp. are) in  $\bigcup_{i \in I} B_i$ .
- $\bar{S}_1^{\text{sac}}$  (resp.  $\bar{S}_2^{\text{sac}}$ ): the set of elements of  $A$  of the type (sac), where one element (resp. two elements) from the triple  $(x, y, z)$  is (resp. are) in  $\mathfrak{B}(A)$ .
- $S_1^{\text{as}}$  (resp.  $S_2^{\text{as}}$ ): the set of elements of  $A$  of the type (as), where one element (resp. two elements) from the triple  $(x, y, z)$  is (resp. are) in  $\bigcup_{i \in I} B_i$ .
- $\bar{S}_1^{\text{as}}$  (resp.  $\bar{S}_2^{\text{as}}$ ): the set of elements of  $A$  of the type (as), where one element (resp. two elements) from the triple  $(x, y, z)$  is (resp. are) in  $\mathfrak{B}(A)$ .
- $S_1^{\text{aas}}$  (resp.  $S_2^{\text{aas}}$ ): the set of elements of  $A$  of the type (aas), where one element (resp. two elements) from the triple  $(x, y, z)$  is (resp. are) in  $\bigcup_{i \in I} B_i$ .
- $\bar{S}_1^{\text{aas}}$  (resp.  $\bar{S}_2^{\text{aas}}$ ): the set of elements of  $A$  of the type (aas), where one element (resp. two elements) from the triple  $(x, y, z)$  is (resp. are) in  $\mathfrak{B}(A)$ .
- $S_1^{\text{ap}}$  (resp.  $S_2^{\text{ap}}$ ): the set of elements of  $A$  of the type (ap), where one element (resp. two elements) from the triple  $(x, y, z)$  is (resp. are) in  $\bigcup_{i \in I} B_i$ .
- $\bar{S}_1^{\text{ap}}$  (resp.  $\bar{S}_2^{\text{ap}}$ ): the set of elements of  $A$  of the type (ap), where one element (resp. two elements) from the triple  $(x, y, z)$  is (resp. are) in  $\mathfrak{B}(A)$ .

**Definition 4.1.** For any  $A \in \text{gAlt}$  define  $\text{Soci}(A)$  as the  $\mathfrak{B}(A)$ -substructure of  $A$  generated by the set  $S_1^{\text{sac}}$ .

**Definition 4.2.** For any  $n \geq 1$  denote  $\text{Asoci}^n(A) = \{x \in A \mid (\dots((xa_1)a_2)\dots a_n) \in \text{Soci}(A), \text{ for any } a_1, a_2, \dots, a_n \in A\}$ . Define  $\text{Asoci}(A) = \bigcup_{n \geq 1} \text{Asoci}^n(A)$ .

From the definition it follows that  $\text{Soci}(A) \subseteq \text{Asoci}^1(A)$  and  $\text{Asoci}^n(A) \subseteq \text{Asoci}^{n+1}(A)$  for any  $n$ .

**Lemma 4.3.** *Let  $A'$  be a  $B_i$ -substructure of  $A$  for any  $i \in I$ . Then  $A'$  is a  $\mathfrak{B}(A)$ -substructure of  $A$  in  $\text{gAlt}_G$ .*

*Proof.* Let  $b = b_i b_j$  be an element of  $\mathfrak{B}(A)$ . For any  $x \in A'$  we have

$$(b_i b_j)x = b_i(b_j x) + b_i(x b_j) - (b_i x)b_j.$$

By the condition of the lemma, every element on the right side is an element of  $A'$ , therefore  $(b_i b_j)x \in A'$ . By induction, in analogous way, it can be proved that  $bx \in A'$  for any  $b \in \mathfrak{B}(A)$ .  $\square$

**Lemma 4.4.** *Asoci( $A$ ) is a two sided ideal of  $A$ .*

*Proof.* It is obvious that if  $x \in \text{Asoci}(A)$ , then  $xa \in \text{Asoci}(A)$  for any  $a \in A$ . Now we shall show that  $ax \in \text{Asoci}(A)$ . Since  $x \in \text{Asoci}(A)$ , there exists a natural number  $k$ , such that for any  $a_1, a_2, \dots, a_k \in A$ ,  $(\dots((-xa_1)a_2)\dots)a_k \in \text{Soci}(A)$ ; in particular,  $(\dots((-xa)a_1)\dots)a_{k-1} \in \text{Soci}(A)$ . We have  $(ax)a_1 \simeq (-xa)a_1$ . By the definition  $\text{Soci}(A)$  is an ideal of  $A$ , therefore  $(\dots((-xa)a_1)\dots)a_{k-1} \simeq (\dots((ax)a_1)\dots)a_{k-1}$ . From this it follows that  $(\dots((ax)a_1)\dots)a_{k-1} \in \text{Soci}(A)$ , which ends the proof.  $\square$

**Proposition 4.5.** *For any  $A \in \text{gAlt}$ , Asoci( $A$ ) is a  $\mathfrak{B}(A)$ -substructure of  $A$ .*

*Proof.* By Lemma 4.3 we have only to prove that for any  $b_i \in B_i$ ,  $i \in I$ , if  $x \in \text{Asoci}(A)$ , then  $xb_i, b_i x \in \text{Asoci}(A)$ . First consider the case, where  $x \in \text{Asoci}^1(A)$ . For any  $a \in A$  we have

$$(xb_i)a = x(b_i a) + x(ab_i) - (xa)b_i.$$

The sum of the first two summands on the right side is in  $\text{Soci}(A)$ ; since  $xa_1 \in \text{Soci}(A)$  and  $\text{Soci}(A)$ , by definition, is a  $\mathfrak{B}(A)$ -substructure in  $A$ , the third summand  $(xa)b_i$  is in  $\text{Soci}(A)$  as well. From this we conclude that  $(xb_i)a \in \text{Soci}(A)$ , which means that  $(xb_i) \in \text{Asoci}(A)$ .

Consider the case where  $x \in \text{Asoci}^2(A)$ . We have  $(xa_1)a_2 \in \text{Soci}(A)$ , for any  $a_1, a_2 \in A$ . We compute

$$((xb_i)a_1)a_2 = (x(b_i a_1) + x(a_1 b_i))a_2 - ((xa_1)b_i)a_2.$$

By Lemma 4.4,  $xa_1 \in \text{Asoci}^1(A)$ , thus from the above proved case we obtain  $(xa_1)b_i \in \text{Soci}(A)$ , and therefore  $((xa_1)b_i)a_2$  is in  $\text{Soci}(A)$  as well. Moreover  $(x(b_i a_1) + x(a_1 b_i))a_2 \in \text{Soci}(A)$  since  $\text{Soci}(A)$  is an ideal of  $A$ . From this we conclude that  $((xb_i)a_1)a_2 \in \text{Soci}(A)$ . In this way by induction we show that for any natural number  $n$  and  $x \in \text{Asoci}^n(A)$ ,  $xb_i \in \text{Asoci}(A)$ . In analogous way is proved that  $b_i x \in \text{Asoci}(A)$ , which ends the proof.  $\square$

**Notation.** For any set  $X$  of elements of  $A$ ,  $A \in \text{gAlt}$ , we will write  $X \simeq 0$  if and only if  $X \subseteq \text{Asoci}(A)$ . We will write  $x_1 \simeq x_2$  (resp.  $x_1 \sim x_2$ ), for  $x_1, x_2 \in A$ , if and only if  $x_1 - x_2 \in \text{Asoci}(A)$  (resp.  $x_1 - x_2 \in \text{Soci}(A)$ ). Therefore  $x_1 \sim x_2$  implies  $x_1 \simeq x_2$ .

By Proposition 4.5 and the definition of  $\text{Soci}(A)$ ,  $\simeq$  and  $\sim$  are congruence relations for the elements in  $A$ .

**Lemma 4.6.** (i)  $S_1^{as} \sim 0$  and  $S_1^{as} \simeq 0$ .

(ii) If  $A$  is an anticommutative  $g$ -alternative algebra, then  $A$  is antiassociative and second level associative, i.e.  $((xy)z)a = (x(yz))a$ , for any  $a, x, y, z \in A$ .

- (iii) If  $A$  is an anticommutative  $g$ -alternative algebra and  $\text{Ann}(A) = 0$ , then  $A$  is anti-associative, associative and  $2x(yz) = 2(xy)z = 0$ , for any  $x, y, z \in A$ .
- (iv) If  $A$  is anticommutative  $g$ -alternative algebra over a field  $F$  with  $\text{char } F \neq 2$  and  $\text{Ann}(A) = 0$ , then  $A = 0$ .

*Proof.* (i) First we show that  $S_1^{aas} \sim 0$ . We have  $(xy)z + x(yz) \sim -(xz)y - y(xz) \sim -(xz)y + (xz)y = 0$ .

For  $S_1^{as} \simeq 0$ , from Axiom 2<sub>1</sub> we obtain  $(x(yz))a \sim -(xa)(yz)$ .

Applying antiassociativity up to congruence relation (i.e. the fact that  $S_1^{aas} \sim 0$ ) and the definition of  $\text{Soci}(A)$  we obtain  $((xy)z)a \sim -((xy)a)z \sim ((xa)y)z \sim -(xa)(yz)$ . Therefore we have  $(x(yz))a - ((xy)z)a \sim 0$ , which proves that  $S_1^{as} \simeq 0$ .

(ii) Antiassociativity of  $A$  is a special case of  $S_1^{aas} \sim 0$  in (i), where “ $\sim$ ” is replaced by “ $=$ ”, since  $A$  is anticommutative. The proof of the second level associativity of  $A$  is a special case of the proof of  $S_1^{as} \simeq 0$  in (i), where  $x, y, z \in A$  and “ $\sim$ ” is replaced by “ $=$ ” again.

(iii) Since  $\text{Ann}(A) = 0$ , second level associativity of  $A$  implies associativity and therefore, applying (ii) we have  $x(yz) = (xy)z = -x(yz)$ , from which follows the result.

(iv) Since from (iii)  $(2 \cdot 1_F)x(yz) = 0$  and  $\text{char } F \neq 2$ , it follows that  $x(yz) = 0$  for any  $x, y, z \in A$ , which implies that  $z = 0$ , for any  $z \in A$ , since  $\text{Ann}(A) = 0$ .  $\square$

**Proposition 4.7.**  $\bar{S}_2^{\text{sac}} \simeq 0$ .

The proof is based on several lemmas.

**Lemma 4.8.**  $S_2^{\text{sac}} \simeq 0$ .

*Proof.* We shall prove the congruence relation for the elements of the type  $x(yz) + x(z y)$ , and the case  $(yz)x + (zy)x$  is left to the reader.

Consider the case where  $x = b_i, y = b_j$ . For any  $a \in A$  we have

$$(b_i(b_j z) + b_i(z b_j))a \sim -(b_i a)(b_j z) - (b_i a)(z b_j).$$

The right side is an element of  $S_1^{\text{sac}}$  and therefore it is an element of  $\text{Soci}(A)$ , which proves that  $(b_i(b_j z) + b_i(z b_j)) \sim 0$ . Analogously, it can be proved that  $((x b_i) b_j + (b_i x) b_j)a \sim 0$  and therefore  $(x b_i) b_j + (b_i x) b_j \simeq 0$ .

Consider the case  $y = b_i, z = b_j$ . Thus we have to show that  $x(b_i b_j) + x(b_j b_i) \in \text{Asoci}(A)$ . For any  $a \in A$  we have  $(x(b_i b_j) + x(b_j b_i))a = ((x b_i) b_j)a + ((b_i x) b_j)a - (b_i(x b_j))a + ((x b_j) b_i)a + ((b_j x) b_i)a - (b_j(x b_i))a$ . Applying the case noted above (i.e.,  $((x b_i) b_j + (b_i x) b_j)a \sim 0$ ) we obtain that the right side is  $\sim$ -congruent to the following one  $-(b_i(x b_j))a - (b_j(x b_i))a = -b_i((x b_j)a) - b_i(a(x b_j)) + (b_i a)(x b_j) - b_j((x b_i)a) - b_j(a(x b_i)) + (b_j a)(x b_i) \sim (b_i a)(x b_j) + (b_j a)(x b_i)$ .

For any  $a' \in A$  we have  $((b_i a)(x b_j))a' \sim -((b_i a)a')(x b_j) \sim (((b_i a')a)x b_j) \sim -(b_i a')(a(x b_j)) \sim (b_i a')(x(a b_j)) \sim -(b_i a')(x(b_j a)) \sim (b_i a')((b_j a)x) \sim -((b_i a')(b_j a))x \sim ((b_j a)(b_i a'))x \sim -(b_j a)((b_i a')x) \sim (b_j a)((b_i x)a') \simeq ((b_j a)(b_i x))a' \sim ((b_j a)(x b_i))a'$ .

Here we applied the fact that  $\text{Soci}(A) \sim 0$  and Lemma 4.6 (i). Thus we obtain that  $((b_i a)(x b_j) + (b_j a)(x b_i))a' \simeq 0$ , which gives that  $((x(b_i b_j) + x(b_j b_i))a)a' \in \text{Asoci}(A)$ , for any  $a, a' \in A$  and therefore  $x(b_i b_j) + x(b_j b_i) \in \text{Asoci}(A)$ .  $\square$

**Lemma 4.9.**  $\bar{S}_1^{\text{as}} \simeq 0$ .

*Proof.* We shall show that  $(b_i b_j)(yz) + (b_i b_j)(zy) \in \text{Asoci}(A)$  and other cases are left to the reader.  $(b_i b_j)(yz) + (b_i b_j)(zy) = b_i(b_j(yz)) + b_i((yz)b_j) - (b_i(yz))b_j + b_i(b_j(zy)) + b_i((zy)b_j) - (b_i(zy))b_j \simeq -(b_i(yz))b_j - (b_i(zy))b_j \simeq (b_i(zy))b_j - (b_i(zy))b_j \simeq 0$ .

Here we applied Proposition 4.5 and Lemma 4.8.  $\square$

**Lemma 4.10.**  $S_2^{\text{as}} \simeq 0, \quad S_2^{\text{aas}} \simeq 0$ .

*Proof.* We shall prove that  $b_k(yb_l) - (b_k y)b_l \simeq 0$ , for any  $y \in A$ ,  $b_k \in B_k$  and  $b_l \in B_l$   $k, l \in I$ ; other cases are proved in analogous ways applying Lemma 4.9. We have

$$(b_k(yb_l))a - ((b_k y)b_l)a \simeq -(b_k a)(yb_l) + ((b_k y)a)b_l \simeq ((b_k a)y)b_l + ((b_k y)a)b_l \simeq -((b_k y)a)b_l + ((b_k y)a)b_l \simeq 0. \quad \square$$

Applying these lemmas we prove Proposition 4.7.

*Proof.* (Proposition 4.7) We shall show that  $(b_i b_j)(y(b_k b_l)) + (b_i b_j)((b_k b_l)y) \in \text{Asoci}(A)$ .

We have  $(b_i b_j)(y(b_k b_l)) + (b_i b_j)((b_k b_l)y) = (b_i b_j)((yb_k)b_l + (b_k y)b_l - b_k(yb_l)) + (b_i b_j)(b_k(b_l y) + b_k(yb_l) - (b_k y)b_l)$ .

Applying Proposition 4.5 and Lemma 4.8 we obtain that this expression is  $\simeq$ -congruent to the following  $-(b_i b_j)(b_k(yb_l)) - (b_i b_j)((b_k y)b_l)$ , which by Lemma 4.10 is  $\simeq$ -congruent to 0.  $\square$

**Lemma 4.11.**  $\bar{S}_1^{\text{as}} \simeq 0, \quad \bar{S}_1^{\text{aas}} \simeq 0$ .

*Proof.* We shall prove that  $((b_i b_j)y)z - (b_i b_j)(yz) \simeq 0$ ,  $y, z \in A$ . We have

$$((b_i b_j)y)z - (b_i b_j)(yz) = (b_i(b_j y) + b_i(yb_j) - (b_i y)b_j)z - b_i(b_j(yz)) - b_i((yz)b_j) + (b_i(yz))b_j \simeq -((b_i y)b_j)z + (b_i(yz))b_j \simeq -((b_i y)b_j)z - (y(b_i z))b_j \simeq ((b_i y)z)b_j - (y(b_i z))b_j \simeq -((yb_i)z)b_j - (y(b_i z))b_j \simeq -(y(b_i z))b_j - (y(b_i z))b_j = 0.$$

Other cases are proved in similar ways applying Proposition 4.7.  $\square$

**Proposition 4.12.**  $\bar{S}_2^{\text{as}} \simeq 0$ .

*Proof.* We shall show that  $(b_i b_j)(y(b_k b_l)) - ((b_i b_j)y)(b_k b_l) \simeq 0$ . The general case can be proved by application Lemmas 4.10 and 4.11. We apply Lemma 4.10 and obtain

$$(b_i b_j)(y(b_k b_l)) - ((b_i b_j)y)(b_k b_l) \simeq (b_i b_j)((yb_k)b_l) - ((b_i b_j)y)b_k b_l \simeq b_i(b_j((yb_k)b_l)) - ((b_i(b_j y))b_k)b_l \simeq b_i((b_j(yb_k))b_l) - ((b_i(b_j y))b_k)b_l \simeq (b_i(b_j(yb_k)))b_l - ((b_i(b_j y))b_k)b_l \simeq (b_i(b_j(yb_k)))b_l - ((b_i(b_j y))b_k)b_l \simeq ((b_i(b_j y))b_k)b_l - ((b_i(b_j y))b_k)b_l \simeq 0. \quad \square$$

In analogous ways are proved the following statements.

**Proposition 4.13.**  $\bar{S}_2^{\text{aas}} \simeq 0$ .

**Lemma 4.14.**  $\bar{S}_1^{\text{ap}} \simeq 0$ .

**Proposition 4.15.**  $\bar{S}_2^{\text{ap}} \simeq 0$ .

5. SUFFICIENT CONDITIONS FOR  $\mathfrak{B}(A) \in \mathbf{gAlt}$ 

We know from Section 2 that  $\mathfrak{B}(A)$  is an object in  $\mathbf{gAlt}_G$ . We begin this section with an example, which shows that the elements of  $\mathfrak{B}(A)$  generally do not satisfy Axiom 2<sub>1</sub> and Axiom 2<sub>2</sub>, and therefore  $\mathfrak{B}(A)$  is not a g-alternative algebra in general.

**Example 5.1.** Let  $A$  and  $\Lambda$  be associative algebras over a field  $F$  with  $\text{char } F \neq 2, 3$ ; let  $A$  be anticommutative,  $\text{Ann}(A) \neq 0$ , and  $A$  has a derived action of  $\Lambda$  in the category of associative algebras, such that  $\lambda aa' = 0$ ,  $\lambda \lambda' a = 0$  and  $\lambda a = -a\lambda$ , for any  $\lambda, \lambda' \in \Lambda$ , and  $a, a' \in A$ . For example one can take  $\Lambda = A$ , since anticommutativity of  $A$  implies antiassociativity, and together with  $\text{char } F \neq 2$  it gives that  $aa'a'' = 0$  for any  $a, a', a'' \in A$ . Let  $R$  be a g-alternative algebra with unit 1, which acts on  $A$  in  $\mathbf{gAlt}$ , in such a way that  $1a = a1 = a$  for any  $a \in A$ .

Let  $A \times A$  be the product associative algebra. Consider the following actions of  $R$  and  $\Lambda$  on  $A \times A$ :

$$r(a, a') = (ra, 0), \quad (a, a')r = (ar, 0), \quad \lambda(a, a') = (0, \lambda a), \quad (a, a')\lambda = (0, a\lambda),$$

for any  $r \in R, \lambda \in \Lambda$  and  $(a, a') \in A \times A$ . It is obvious that the action of  $R$  on  $A \times A$  is a derived action and, it can be easily checked, that the action of  $\Lambda$  on  $A \times A$  is a derived action as well. We will show that the equality

$$b_i(b_j b_k) = (b_i b_j)b_k + (b_j b_i)b_k - b_j(b_i b_k),$$

where  $b_i \in B_i, b_j \in B_j, b_k \in B_k$  doesn't hold in general.

Consider the case, where  $b_i = \lambda, b_j = b_k = 1, \lambda \in \Lambda$ , and 1 is unit of  $R$ .

We first compute the results of the following actions and obtain:

$$\begin{aligned} (1\lambda)(a, a') &= -(0, a\lambda), & (\lambda 1)(a, a') &= (0, 2\lambda a), \\ (a, a')(1\lambda) &= (0, 2a\lambda), & (a, a')(\lambda 1) &= -(0, \lambda a), \\ \lambda(1(a, a')) &= (0, \lambda a), & \lambda((a, a')1) &= (0, \lambda a), \\ (1(a, a'))\lambda &= (0, a\lambda), & ((a, a')1)\lambda &= (0, a\lambda), \\ r((a, a')\lambda) &= (0, 0), & r(\lambda(a, a')) &= (0, 0), \\ ((a, a')\lambda)r &= (0, 0), & (\lambda(a, a'))r &= (0, 0), \end{aligned}$$

for any  $(a, a') \in A \times A, r \in R$  and unit 1 of  $R$ .

We shall show that the following equality is not true in general

$$(\lambda(1 \cdot 1))(a, a') = ((\lambda 1)1)(a, a') + ((1\lambda)1)(a, a') - (1(\lambda 1))(a, a').$$

The computations of both sides give that this equality is equivalent to the following one  $2\lambda a = 4\lambda a - 2a\lambda - \lambda a = 0$ .

Since  $\lambda a = -a\lambda$  and  $\text{char } F \neq 3$ , this equality gives  $\lambda a = 0$ , which is not true in general.

This shows that in the case of this example Axiom 2<sub>1</sub> is not true. The same example Axiom 2<sub>2</sub> is not true as well. This can be checked by analogous computations or we can apply the duality in the following way. Define in  $A$  the dual operation by  $x \circ y = yx$ . Axiom 2<sub>2</sub> for the original dot operation is equivalent to the Axiom 2<sub>1</sub> for the dual “ $\circ$ ”

operation. But since both operations have the same properties, we can conclude from the above prove that Axiom 2<sub>1</sub> is not true for the “o” operation.

We are looking for the sufficient conditions for  $\mathfrak{B}(A)$  to be a g-alternative algebra, i.e. for the conditions under which the elements of  $\mathfrak{B}(A)$  satisfy Axioms 2<sub>1</sub> and 2<sub>2</sub>. For any  $b_1, b_2, b_3 \in \mathfrak{B}(A)$  we must have the following identities

$$\begin{aligned} \text{B1.} \quad & -(b_1(b_2b_3))a + ((b_1b_2)b_3)a + ((b_2b_1)b_3)a - (b_2(b_1b_3))a = 0, \\ \text{B2.} \quad & -a(b_1(b_2b_3)) + a((b_1b_2)b_3) + a((b_2b_1)b_3) - a(b_2(b_1b_3)) = 0; \end{aligned}$$

and the dual identities

$$\begin{aligned} \text{B3} &= \text{B2}^\circ. \quad -((b_1b_2)b_3)a + (b_1(b_2b_3))a + (b_1(b_3b_2))a - ((b_1b_3)b_2)a = 0, \\ \text{B4} &= \text{B1}^\circ. \quad -a((b_1b_2)b_3) + a(b_1(b_2b_3)) + a(b_1(b_3b_2)) - a((b_1b_3)b_2) = 0. \end{aligned}$$

First we compute the left side of identity B1. By the definition of the multiplication in  $\mathfrak{B}(A)$  we obtain:

$$\begin{aligned} (5.1) \quad & -(b_1(b_2b_3))a + ((b_1b_2)b_3)a + ((b_2b_1)b_3)a - (b_2(b_1b_3))a \\ &= -b_1((b_2b_3)a) - b_1(a(b_2b_3)) + (b_1a)(b_2b_3) + (b_1b_2)(b_3a) + (b_1b_2)(ab_3) - ((b_1b_2)a)b_3 \\ & \quad + (b_2b_1)(b_3a) + (b_2b_1)(ab_3) - ((b_2b_1)a)b_3 - b_2((b_1b_3)a) - b_2(a(b_1b_3)) + (b_2a)(b_1b_3) \\ &= -b_1(b_2(b_3a)) - b_1(b_2(ab_3)) + b_1((b_2a)b_3) - b_1((ab_2)b_3) - b_1((b_2a)b_3) + b_1(b_2(ab_3)) \\ & \quad + ((b_1a)b_2)b_3 + (b_2(b_1a))b_3 - b_2((b_1a)b_3) + b_1(b_2(b_3a)) + b_1((b_3a)b_2) - (b_1(b_3a))b_2 \\ & \quad + b_1(b_2(ab_3)) + b_1((ab_3)b_2) - (b_1(ab_3))b_2 - (b_1(b_2a))b_3 - (b_1(ab_2))b_3 + ((b_1a)b_2)b_3 \\ & \quad + b_2(b_1(b_3a)) + b_2((b_3a)b_1) - (b_2(b_3a))b_1 + b_2(b_1(ab_3)) + b_2((ab_3)b_1) - (b_2(ab_3))b_1 \\ & \quad - (b_2(b_1a))b_3 - (b_2(ab_1))b_3 + ((b_2a)b_1)b_3 - b_2(b_1(b_3a)) - b_2(b_1(ab_3)) + b_2((b_1a)b_3) \\ & \quad - b_2((ab_1)b_3) - b_2((b_1a)b_3) + b_2(b_1(ab_3)) + ((b_2a)b_1)b_3 + (b_1(b_2a))b_3 - b_1((b_2a)b_3). \end{aligned}$$

The left side of identity B2 gives the following

$$\begin{aligned} (5.2) \quad & -a(b_1(b_2b_3)) + a((b_1b_2)b_3) + a((b_2b_1)b_3) - a(b_2(b_1b_3)) \\ &= -(ab_1)(b_2b_3) - (b_1a)(b_2b_3) + b_1((ab_2)b_3) - ((b_1a)b_2)b_3 - (b_1(b_2a))b_3 + b_1((b_2a)b_3) \\ & \quad + b_2((ab_1)b_3) + b_2((b_1a)b_3) - b_2(b_1(ab_3)) - (ab_2)(b_1b_3) - (b_2a)(b_1b_3) + b_2(a(b_1b_3)) \\ &= -((ab_1)b_2)b_3 - (b_2(ab_1))b_3 + b_2((ab_1)b_3) - ((b_1a)b_2)b_3 - (b_2(b_1a))b_3 + b_2((b_1a)b_3) \\ & \quad + b_1((ab_2)b_3) + b_1((b_2a)b_3) - b_1(b_2(ab_3)) + ((ab_1)b_2)b_3 + ((b_1a)b_2)b_3 - (b_1(ab_2))b_3 \\ & \quad + (b_1(b_2a))b_3 + (b_1(ab_2))b_3 - ((b_1a)b_2)b_3 - b_1(b_2(ab_3)) - b_1((ab_3)b_2) + (b_1(ab_3))b_2 \\ & \quad + ((ab_2)b_1)b_3 + ((b_2a)b_1)b_3 - (b_2(ab_1))b_3 + (b_2(b_1a))b_3 + (b_2(ab_1))b_3 - ((b_2a)b_1)b_3 \\ & \quad - b_2(b_1(ab_3)) - b_2((ab_3)b_1) + (b_2(ab_3))b_1 - ((ab_2)b_1)b_3 - (b_1(ab_2))b_3 + b_1((ab_2)b_3) \\ & \quad - ((b_2a)b_1)b_3 - (b_1(b_2a))b_3 + b_1((b_2a)b_3) + b_2((ab_1)b_3) + b_2((b_1a)b_3) - b_2(b_1(ab_3)). \end{aligned}$$

The identities B3 and B4 give the duals to the expressions (5.2) and (5.1) respectively.

It is easy to see that all obtained expressions are the combinations of the elements of the following type

$$\begin{aligned}
\mathbf{A1} &= b_1(b_2(b_3a)) + b_1(b_2(ab_3)), \\
\mathbf{A2} &= b_1((ab_2)b_3) + b_1((b_2a)b_3), \\
\mathbf{A3} &= ((b_1a)b_2)b_3 + (b_2(b_1a))b_3, \\
\mathbf{A4} &= b_1(b_2(ab_3)) + b_2(b_1(ab_3)), \\
\mathbf{A5} &= ((b_1a)b_2)b_3 + ((b_2a)b_1)b_3, \\
\mathbf{A6} &= b_1(b_2(b_3a)) + b_1((b_3a)b_2), \\
\mathbf{A7} &= b_1(b_2(ab_3)) + b_1((ab_3)b_2), \\
\mathbf{A8} &= (b_1(b_2a))b_3 + (b_1(ab_2))b_3, \\
\mathbf{A9} &= ((ab_1)b_2)b_3 + (b_2(ab_1))b_3, \\
\mathbf{A10} &= ((ab_3)b_1)b_2 + ((ab_3)b_2)b_1, \\
\mathbf{A11} &= b_3(b_2(ab_1)) + b_3(b_1(ab_2)),
\end{aligned}$$

where  $b_1, b_2, b_3 \in \mathfrak{B}(A), a \in A$ .

**Theorem 5.1.** *If for any  $b_1, b_2, b_3 \in \mathfrak{B}(A), a \in A$ , we have the equalities  $\mathbf{A}i = 0$  for  $i = 1, \dots, 11$ , then  $\mathfrak{B}(A)$  is a  $g$ -alternative algebra.*

*Proof.* The proof follows directly from the identities (5.1), (5.2) and the dual identities, which proof that under the conditions of the theorem we have Axioms 2<sub>1</sub> and 2<sub>2</sub> for the elements of  $\mathfrak{B}(A)$ .  $\square$

On the other hand the following proposition shows that  $\mathfrak{B}(A)$  is a  $g$ -alternative algebra up to the congruence relation  $\simeq$ .

**Proposition 5.2.** *For any  $b_1, b_2, b_3 \in \mathfrak{B}(A), a \in A$ , we have  $\mathbf{A}i \simeq 0$  for  $i = 1, \dots, 11$ .*

*Proof.* Direct application of the statements 4.7, 4.12, 4.13 and 4.15.  $\square$

**Corollary 5.3.** *If  $\text{Asoci}(A) = 0$ , then  $\mathfrak{B}(A)$  is a  $g$ -alternative algebra.*

**Lemma 5.4.** *Let  $\{B_i\}_{i \in I}$  be the family of all  $g$ -alternative algebras which have a derived action on  $A$ . The following conditions are equivalent:*

- (a)  $\text{Asoci}(A) = 0$ ;
- (b)  $\text{Asoci}^1(A) = 0$ ;
- (c) *every derived action of  $B_i$  on  $A$  is anticommutative, for any  $i \in I$  (i.e.  $b_i a = -ab_i$ ,  $b_i \in B_i, a \in A$ ) and  $\text{Ann}(A) = 0$ .*

*Proof.* The implication (a) $\Rightarrow$ (b) is obvious, since  $\text{Asoci}^1(A) \subseteq \text{Asoci}(A)$ .

(b) $\Rightarrow$ (c). We have  $b_i a + ab_i \in \text{Asoci}^1(A)$ ; since  $\text{Asoci}^1(A) = 0$ , it follows that every derived action is anticommutative; in particular, since  $A$  has a derived action on itself,  $A$  is anticommutative. From this it follows that the right and left annihilators of  $A$  coincide and thus  $\text{rAnn}(A) = \text{lAnn}(A) = \text{Ann}(A)$ . As we have noted in Section 4, we have an inclusion  $\text{Soci}(A) \subseteq \text{Asoci}^1(A)$ , from which we obtain that  $\text{Soci}(A) = 0$ . Therefore we obtain that  $\text{Ann}(A) = \text{Asoci}^1(A) = 0$ .

(c) $\Rightarrow$ (a). Since every action of  $B_i$  on  $A$  is anticommutative, we have  $\text{Soci}(A) = 0$ . Therefore for  $x \in \text{Asoci}(A)$  there exists a natural number  $n$ , such that for any  $a_1, \dots, a_n \in A$  we have  $(\dots((xa_1)a_2)\dots a_n) = 0$ . Since  $\text{Ann}(A) = 0$ , it follows that  $(\dots((xa_1)a_2)\dots a_{n-1}) =$



0. Thus applying analogous arguments we obtain that  $xa_1 = 0$  for any  $a_1 \in A$  and therefore  $x = 0$  since  $\text{Ann}(A) = 0$ , which ends the proof.  $\square$

**Proposition 5.5.** *If  $A$  is anticommutative  $g$ -alternative algebra over a field  $F$  and  $\text{Ann}(A) = 0$ , then  $(a_1b)a_2 = a_1(ba_2)$ , for any  $g$ -alternative algebra  $B$  with derived action on  $A$  and any  $a_1, a_2 \in A, b \in B$ .*

*Proof.* If  $\text{char } F \neq 2$ , then by Lemma 4.6 (iv)  $A = 0$ , so the equality always holds. Consider the case  $\text{char } F = 2$ . Since  $A$  is anticommutative, by Lemma 4.6 (ii), it is antiassociative as well, and therefore, for any  $a \in A$ , we have the following equalities

$$\begin{aligned} ((a_1b)a_2)a &= -(a_2(a_1b))a = (a_1(a_2b))a = -(a_1a)(a_2b) = a_2((a_1a)b) = \\ &= -((a_1a)b)a_2 = -(a_1(ab) + a_1(ba) - (a_1b)a)a_2 = -(-a(a_1b) + a_1(ba) - (a_1b)a)a_2 = \\ &= -((a_1b)a + a_1(ba) - (a_1b)a)a_2 = -(a_1(ba))a_2 = a_1((ba)a_2) = -a_1((ba_2)a) = (a_1(ba_2))a, \\ &= (a_1(ba_2))a = -(a_1a)(ba_2) = (a_1(ba_2))a, \end{aligned}$$

from which by the condition  $\text{Ann}(A) = 0$  follows the desired equality.  $\square$

Note that the proof of this proposition doesn't follow from Lemma 4.6 (i) by taking there  $x = a_1, y = b, z = a_2$ , since in the proof of  $S_1^{\text{as}} \sim 0$  is involved  $\text{Soci}(A)$ , which contains an element from  $B$ .

**Proposition 5.6.** *Let  $A$  be a  $g$ -alternative algebra with  $\text{Ann}(A) = 0$ . The following conditions are equivalent:*

- (i) *for any  $g$ -alternative algebra  $B$  with derived action on  $A$  in  $\text{gAlt}$ , we have  $ba = -ab$  for any  $a \in A, b \in B$ ;*
- (ii)  *$A$  is anticommutative.*

*Proof.* Here, as in the previous proposition, we need to prove only the case, where  $\text{char } F = 2$ . It is obvious that (i)  $\Rightarrow$  (ii), since  $A$  has a derived action on itself.

(ii)  $\Rightarrow$  (i). For any  $a', a'' \in A$  we have  
 $a''(a'(ab) + a'(ba)) = a''(-a(a'b) + a'(ba)) = a''(-a(a'b) + (a'b)a + (ba')a - b(a'a)) =$   
 $a''(-a(a'b) - a(a'b) + (ba')a - (ba')a - (a'b)a + a'(ba)) = -2a''(a(a'b)) = 0.$

Here we applied anticommutativity of  $A$ , Axiom 2, Proposition 5.5 and Lemma 4.6 (iii).  $\square$

**Proposition 5.7.** *If  $A$  is an anticommutative  $g$ -alternative algebra and  $\text{Ann}(A) = 0$ , then  $\mathfrak{B}(A)$  is a  $g$ -alternative algebra.*

*Proof.* Apply Corollary 5.3, Lemma 5.4 and Proposition 5.6.  $\square$

## 6. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF AN ACTOR IN $\text{gAlt}$

It is obvious that the action of  $\mathfrak{B}(A)$  on  $A$  satisfies identities  $\text{II}_1$  and  $\text{II}_2$ ; but this action in general is not a derived action. We begin with examples, which show that all other action identities fail in general.

**Example 6.1.** Here we consider the product algebra  $A \times A$  of Example 5.1 and we show that the identity  $\Pi_3$  is not true in general. We take in  $\Pi_3$ : instead of  $a$  the element  $(a, a') \in A \times A$ ,  $b_1 = 1, b_2 = \lambda$  and obtain that in this case  $\Pi_3$  is equivalent to the following equality  $(0, a\lambda) = (0, 2a\lambda) - (0, \lambda a)$ .

From this, since by assumption  $a\lambda = -\lambda a$  and  $\text{char } F \neq 2$ , we obtain that  $a\lambda = 0$ , which is not true in general.

**Example 6.2.** Consider the same example of the algebra  $A \times A$  as in Examples 5.1 and 6.1. We take in  $\Pi_4$  instead of  $a$  the element  $(a, a')$ ,  $b_1 = \lambda, b_2 = 1$  and obtain  $(0, \lambda a) = (0, 2\lambda a) - (0, a\lambda)$ .

As in the previous example, this implies  $a\lambda = 0$ , which is not true in general.

**Example 6.3.** Here we consider the example which shows that the identity  $I_1$  is not always true. Let  $A$  and  $R$  be commutative, associative algebras over a field  $F$  with characteristic 2, and  $R$  has a derived action on  $A$  in the category of commutative, associative algebras, i.e. together with the conditions given in Section 2. We have  $ra = ar$ , for any  $a \in A$  and  $r \in R$ . Obviously this will be a derived action in the category of g-alternative algebras as well. Let  $\Lambda$  be a g-alternative algebra over the same field  $F$ , which acts on  $A$  in  $\text{gAlt}$ , and there exists an element  $a' \in A$ , such that  $a'\lambda$  is not a zero divisor in  $A$ . We shall show that the following equality is not true

$$(r\lambda)(aa') = ((r\lambda)a)a' + (a(r\lambda))a' - a((r\lambda)a'),$$

for any  $a \in A$ . Under the above assumptions on actions and algebras the computations give the following:

- $(r\lambda)(aa') = r(\lambda(aa')) + r((aa')\lambda) - (r(aa'))\lambda = r((\lambda a)a') + r((a\lambda)a') - r(a(\lambda a')) - r((a\lambda)a') + ((ra)\lambda)a'$ ;
- $((r\lambda)a)a' = (r(\lambda a))a' + (r(a\lambda))a' - ((ra)\lambda)a'$ ;
- $(a(r\lambda))a' = -(r(a\lambda))a'$ ;
- $-a((r\lambda)a') = -a(r(\lambda a')) - a(r(a'\lambda)) + a((ra')\lambda)$ .

Thus the above equality is equivalent to the following one:  $a((ra')\lambda) = (ar)(a'\lambda)$ .

Applying the fact that  $A$  is commutative and  $\text{char } F = 2$ , we have  $a((ra')\lambda) = -(ra')(a\lambda)$ ; therefore we obtain  $-(ra')(a\lambda) = (ar)(a'\lambda)$ , which can not be true for any  $a$ , since in this case  $a\lambda = 0$  implies  $ar = 0$ , because  $a'\lambda$  is not a zero divisor by assumption.

The cases of the identities  $I_2, I_3, I_4$  are considered in analogous ways.

**Theorem 6.1.** *If  $\text{Asoci}(A) = 0$ , then the action of  $\mathfrak{B}(A)$  on  $A$  is a derived action.*

*Proof.* From the definition of the multiplication in  $\mathfrak{B}(A)$  it is easy to see that the identities  $\Pi_1$  and  $\Pi_2$  of Section 3 always hold for the action of  $\mathfrak{B}(A)$  on  $A$ . For the identities  $I_1 - I_4$  and  $\Pi_3$  and  $\Pi_4$  we apply statements 4.11 and 4.12, which show that under the condition of the theorem  $\mathfrak{B}(A)$  has a derived action on  $A$ . Note that the same can be proved by the statements 4.9, 4.14, 4.7 and 4.15.  $\square$

**Corollary 6.2.** *If  $\text{Asoci}(A) = 0$ , then  $\mathfrak{B}(A)$  is an actor of  $A$ .*

*Proof.* Apply Corollaries 2.6, 5.3 and Theorem 6.1.  $\square$

**Corollary 6.3.** *If  $A$  is anticommutative  $g$ -alternative algebra over a field  $F$  and  $\text{Ann}(A) = 0$ , then there exists an actor of  $A$  and  $\text{Act}(A) = \mathfrak{B}(A)$ .*

*Proof.* If  $\text{char } F \neq 2$ , then, by Lemma 4.6 (iv), it follows that  $A = 0$ , and obviously  $\text{Act}(A) = \mathfrak{B}(A) = 0$ . If  $\text{char } F = 2$ , we apply Lemma 5.4, Propositions 5.6 and 5.7, Corollary 6.2 and obtain the result.  $\square$

Here we give the construction of an  $F$ -algebra of bimultiplications  $\text{Bim}_{\text{gAlt}}(A)$  of a  $g$ -alternative algebra  $A$  over a field  $F$ . Below is used the notation  $f*$  and  $*f$  for the  $F$ -linear maps  $A \rightarrow A$ ; we will denote by  $fa$  (resp.  $af$ ) the value  $(f*)(a)$  (resp.  $(*f)(a)$ ). This kind of notation (similar to the one of the actions  $b*a$  and  $a*b$  in a category of interest) makes simpler to write down the conditions for bimultiplications; we will see that these conditions are simply Axioms 2<sub>1</sub> and 2<sub>2</sub> written for the four different ordered triples. An element of  $\text{Bim}_{\text{gAlt}}(A)$  is a pair  $f = (f*, *f)$  of  $F$ -linear maps from  $A$  to  $A$ , which satisfies the following conditions

$$(6.1) \quad \begin{aligned} f(a_1a_2) &= (fa_1)a_2 + (a_1f)a_2 - a_1(fa_2), \\ (a_1a_2)f &= a_1(a_2f) + a_1(fa_2) - (a_1f)a_2, \\ (fa_1)a_2 &= f(a_1a_2) + f(a_2a_1) - (fa_2)a_1, \\ a_1(a_2f) &= (a_1a_2)f + (a_2a_1)f - a_2(a_1f). \end{aligned}$$

The product of the elements  $f = (f*, *f)$  and  $f' = (f'*, *f')$  of  $\text{Bim}_{\text{gAlt}}(A)$  is defined by

$$ff' = (f * f'*, *f * f'),$$

here on the right side  $f * f'*$  and  $*f * f'$  are defined by

$$\begin{aligned} (f * f'*)(a) &= f(f'a) + f(af') - (fa)f', \\ (*f * f')(a) &= (af)f' + (fa)f' - f(af'). \end{aligned}$$

For the addition we have

$$f + f' = ((f*) + f'*, *f + (*f')),$$

where

$$\begin{aligned} ((f*) + f'*)(a) &= fa + f'a, \\ (*f + (*f'))(a) &= af + af'. \end{aligned}$$

The product of two bimultiplications may not have the properties (6.1). Therefore we include in  $\text{Bim}_{\text{gAlt}}(A)$  all the new obtained pairs of  $F$ -linear maps. Note that different products can give one and the same pairs of maps, i.e.  $(\varphi*, *\varphi) = (\varphi'*, *\varphi')$  if  $\varphi a = \varphi'a$  and  $a\varphi = a\varphi'$ , where  $\varphi = (\varphi*, *\varphi)$  and  $\varphi' = (\varphi'*, *\varphi')$  are certain combinations of bimultiplications.

It is obvious that  $\text{Bim}_{\text{gAlt}}(A)$  is an  $F$ -algebra and it is an object of  $\text{gAlt}_G$  in general. In the same way as Corollary 6.3, it can be proved

**Theorem 6.4.** *Let  $A$  be an anticommutative  $g$ -alternative algebra with  $\text{Ann}(A) = 0$ , then there exists an actor of  $A$  and  $\text{Act}(A) = \text{Bim}_{\text{gAlt}}(A)$ .*

**Corollary 6.5.** *Under the condition of Theorem 6.4 we have  $\text{Bim}_{\text{gAlt}}(A) = \mathfrak{B}(A)$ .*

*Proof.* The conditions of Corollary 6.3 are fulfilled, from which it follows that  $\mathfrak{B}(A)$  is an actor of  $A$ . From Theorem 6.4 and the universal property of an actor we obtain the desired equality.  $\square$

## 7. THE EXISTENCE OF AN ACTOR IN Alt

As we have noted in Section 2, by definition of general category of interest, we have  $\text{Alt}_G = \text{gAlt}_G$ . Let  $\text{Alt}$  be a category of alternative algebras over a field  $F$  with  $\text{char } F = 2$ . Then, by Proposition 2.8,  $\text{Alt} \subset \text{gAlt}$ . In the categories  $\text{Alt}$  and  $\text{gAlt}$  Axiom 2 is the same, according to which the multiplication in the construction of a universal strict general actor is defined. Therefore, for any  $A \in \text{Alt}$ , the algebra  $\mathfrak{B}_{\text{Alt}}(A)$ , constructed for the derived actions in  $\text{Alt}$ , is a subalgebra of  $\mathfrak{B}(A)$ , constructed for the derived actions in  $\text{gAlt}$  for the same alternative algebra  $A$ . Thus we have the inclusion of algebras  $\mathfrak{B}_{\text{Alt}}(A) \subseteq \mathfrak{B}(A)$ .

**Proposition 7.1.** *If  $A$  is anticommutative  $g$ -alternative algebra and  $\text{Ann}(A) = 0$ , then  $\mathfrak{B}(A)$  is also anticommutative and  $\text{Ann}(\mathfrak{B}(A)) = 0$ .*

*Proof.* If  $\text{char } F \neq 2$ , then, by Lemma 4.6 (iv), it follows that  $A = 0$ , and obviously  $\mathfrak{B}(A) = \text{Ann}(\mathfrak{B}(A)) = 0$ . If  $\text{char } F = 2$ , then by Proposition 5.6 every derived action on  $A$  is anticommutative; from this and the condition  $\text{Ann}(A) = 0$ , by Lemma 5.4 it follows that  $\text{Asoci}(A) = 0$ . Applying this fact, from Proposition 4.7, we obtain that  $\bar{S}_2^{\text{sac}} = 0$ , which gives the equalities  $a(b_1b_2) = -a(b_2b_1)$  and  $(b_1b_2)a = -(b_2b_1)a$ , for any  $a \in A$  and  $b_1, b_2 \in \mathfrak{B}(A)$ , which by the construction of  $\mathfrak{B}(A)$ , proves that  $\mathfrak{B}(A)$  is anticommutative. Now we shall prove that  $\text{Ann}(\mathfrak{B}(A)) = 0$ . Suppose  $b_1b = 0$  for any  $b \in \mathfrak{B}(A)$ , which means that  $(b_1b)a = 0$ , for any  $a \in A$ . By the definition of multiplication in  $\mathfrak{B}(A)$  and an action of  $\mathfrak{B}(A)$  on  $A$  we have

$$(b_1b)a = b_1(ba) + b_1(ab) - (b_1a)b = 0.$$

Since the action of  $\mathfrak{B}(A)$  is anticommutative, we obtain that  $(b_1a)b = 0$ , for any  $b \in \mathfrak{B}(A)$ , and in particular, for  $b = a'$ , where  $a'$  is any element from  $A$ . This gives  $(b_1a)a' = 0$ , for any  $a'$ , which means that  $b_1a$  is an annihilator in  $A$ , therefore  $b_1a = 0$ , for any  $a \in A$ , which, by construction of  $\mathfrak{B}(A)$ , implies that  $b_1 = 0$ .  $\square$

Note that in the proof of Proposition 7.1 we could take  $b = \mathbf{a}$  in  $b_1b = 0$ , for any  $a \in A$ . In this case we should prove that  $(b_1\mathbf{a})a' = (b_1a)a'$ , for any  $a' \in A$ . For this we would have to apply the fact that  $\mathfrak{B}(A)$  has a derived action on  $A$  in  $\text{gAlt}$ . From this we would obtain that  $b_1\mathbf{a}$  and  $b_1a$  are equal in  $\mathfrak{B}(A)$ , i.e.  $b_1\mathbf{a} = cl(b_1a)$ , which will imply that  $(b_1a)a' = 0$ . From this, since  $\text{Ann}(A) = 0$ , it would follow that  $b_1a = 0$ , and therefore  $b_1 = 0$ . As we see this proof is longer than we have presented.

It is easy to see, and it is noted in [21, 22], that any commutative or anticommutative algebra satisfies the flexible law  $E_1$ . Therefore, from this proposition and Proposition 2.8 (i), it follows that, under the conditions of Proposition 7.1,  $\mathfrak{B}(A)$  is an alternative algebra. The following corollary proves that  $\mathfrak{B}(A)$  is an associative algebra as well.

**Corollary 7.2.** *If  $A$  is anticommutative  $g$ -alternative algebra with  $\text{Ann}(A) = 0$ , then  $\mathfrak{B}(A)$  is an associative algebra, in particular,  $\mathfrak{B}(A) \in \text{Alt}$ .*

*Proof.* Apply Proposition 7.1 and Lemma 4.6 (iii).  $\square$

**Theorem 7.3.** *If  $A$  is anticommutative alternative algebra over a field  $F$  with  $\text{Ann}(A) = 0$ , then there exists an actor of  $A$  in  $\text{Alt}$  and  $\text{Act}(A) = \mathfrak{B}_{\text{Alt}}(A) = \mathfrak{B}(A)$ .*

*Proof.* By Proposition 2.8, for any field  $F$ , we have  $\text{Alt} \subseteq g\text{Alt}$ . If  $\text{char } F \neq 2$ , then, by Lemma 4.6 (iv), it follows that  $A = 0$ , and obviously  $\text{Act}(A) = \mathfrak{B}(A) = 0$ . Suppose  $\text{char } F = 2$ . Since  $A$  is a  $g$ -alternative algebra, we can apply Corollary 7.2, and therefore  $\mathfrak{B}(A)$  is an alternative algebra. By Corollary 6.3 the action of  $\mathfrak{B}(A)$  on  $A$  is a derived action in  $g\text{Alt}$ . We shall prove that the action of  $\mathfrak{B}(A)$  on  $A$  satisfies the conditions III<sub>1</sub> and III<sub>2</sub> of Section 3, for any  $a \in A$  and  $b \in \mathfrak{B}(A)$ . By Proposition 5.6, every derived action on  $A$  is anticommutative, and by Lemma 5.4, under the conditions of the theorem we obtain  $\text{Asoci}(A) = 0$ ; therefore applying Lemma 4.11 and Proposition 4.12 we obtain that  $\bar{S}_1^{\text{as}} = 0$  and  $\bar{S}_2^{\text{as}} = 0$ , which imply respectively the equalities III<sub>1</sub> and III<sub>2</sub>.  $\mathfrak{B}(A)$  is an actor of  $A$  in  $g\text{Alt}$ , therefore by the definition of an actor we obtain, that  $\mathfrak{B}(A)$  is an actor of  $A$  in  $\text{Alt}$ . Now the result follows from Theorem 2.5.  $\square$

The construction of an  $F$ -algebra of bimultiplications  $\text{Bim}_{\text{Alt}}(A)$  of an alternative algebra  $A$  over a field  $F$  is analogous to  $\text{Bim}_{g\text{Alt}}(A)$ , where in addition we require that pairs  $f = (f*, *f)$  of  $F$ -linear maps from  $A$  to  $A$  together with conditions (6.1) satisfy the following two conditions:

$$\begin{aligned} a(fa) &= (af)a, \\ f(af) &= (fa)f. \end{aligned}$$

The statements analogous to Theorem 6.4 and Corollary 6.5 take place for alternative algebras. Obviously, every commutative associative algebra over a field with characteristic 2 is anticommutative  $g$ -alternative algebra, and every anticommutative  $g$ -alternative algebra  $A$  over the same kind field with  $\text{Ann}(A) = 0$  is associative and commutative (apply Lemma 4.6 (ii)). Let  $\text{Bim}(A)$  denote the algebra of bimultiplications of an associative algebra  $A$  [11]. It is proved in [8] that if  $A$  is any associative algebra with  $\text{Ann}(A) = 0$ , then there exists an actor of  $A$  in the category of associative algebras and  $\text{Act}(A) = \text{Bim}(A)$ . The analogous result we have in the category of commutative associative algebras, under the same condition an actor exists and this is the commutative algebra of multiplications  $M(A)$  [14] (or multipliers [13]) of  $A$ , which is defined as an algebra of  $F$ -linear maps  $f: A \rightarrow A$  with  $f(aa') = f(a)a'$ , for any  $a, a' \in A$ . Let  $A$  be an associative commutative algebra over a field  $F$  with characteristic 2 and  $\text{Ann}(A) = 0$ . The equalities  $\bar{S}_1^{\text{as}} = 0$  and  $\bar{S}_2^{\text{as}} = 0$  in the proof of Theorem 7.3 imply that the action of  $\mathfrak{B}(A)$  on  $A$  is a derived action in the category of associative algebras. At the same time we know that  $\mathfrak{B}(A)$  is anticommutative and its action on  $A$  is anticommutative as well, therefore  $\mathfrak{B}(A)$  is commutative and the action on  $A$  is commutative too. Therefore, in the same way as in the proof of Theorem 7.3, we conclude that  $\mathfrak{B}(A)$  is an actor of  $A$  in the categories of associative and commutative associative algebras. From the universal property of an actor we obtain, that if  $A$  is

commutative associative algebra over a field  $F$  with characteristic 2 and  $\text{Ann}(A) = 0$ , then  $\mathfrak{B}(A) = \text{Bim}(A) = M(A)$ .

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